 The Girard-Reynolds Isomorphism 

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A tale of Two Theorems

Girard's Representation Theorem

Reynolds's Abstraction Theorem

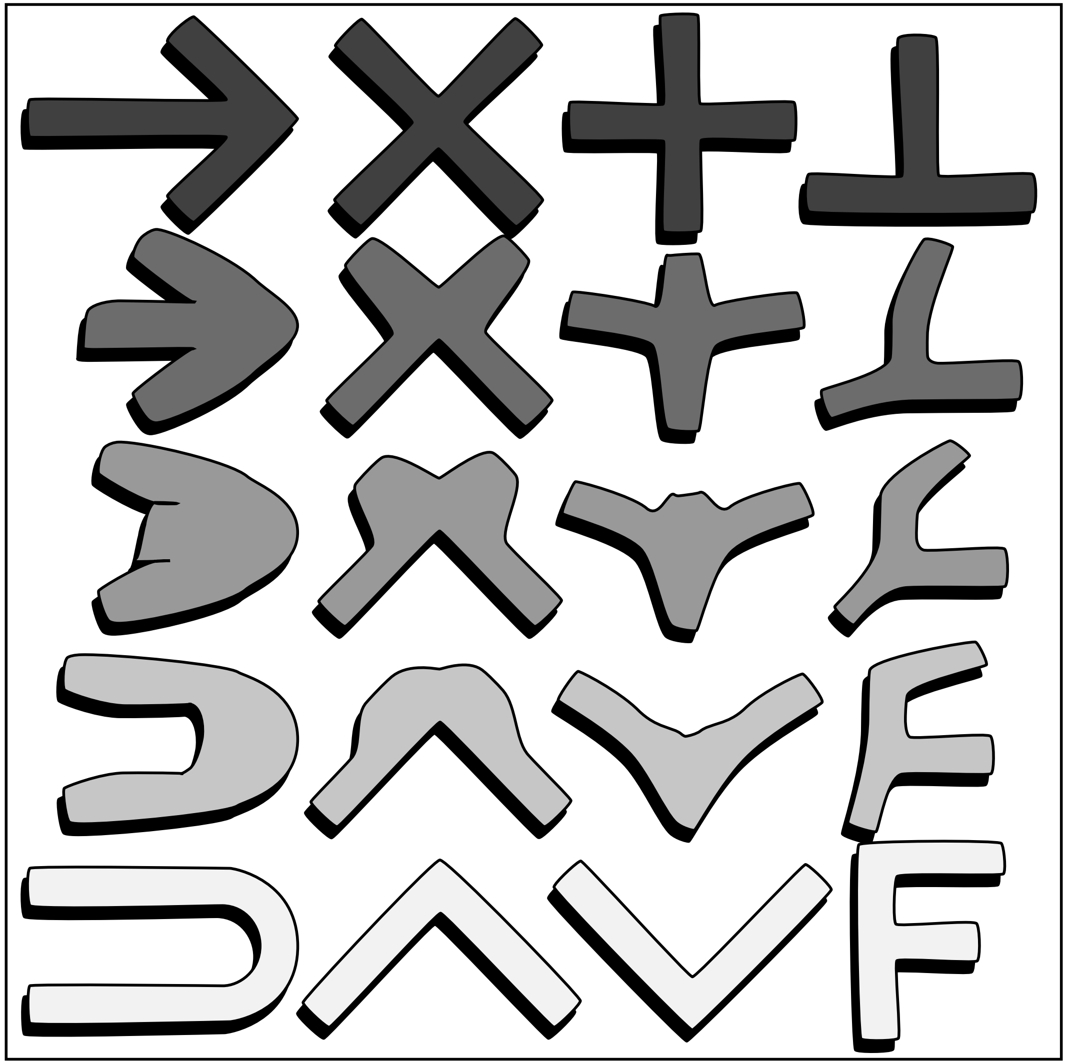
A tale of Two Theorems

Girard's Representation Theorem

projection : proofs $\longrightarrow$ terms

Reynolds's Abstraction Theorem

embedding : terms $\longrightarrow$ proofs



LC'90 *The Curry-Howard homeomorphism*

The Curry-Howard Isomorphism

$$\frac{\Pi \quad \longrightarrow \quad \times \quad + \quad \bot}{\forall \quad \supset \quad \wedge \quad \vee \quad F}$$

The Curry-Howard Isomorphism

$$\frac{\Pi \quad \rightarrow \quad \times \quad + \quad \bot}{\forall \quad \supset \quad \wedge \quad \vee \quad F}$$

The Girard-Reynolds Isomorphism

$$\frac{\forall \quad \forall^2 \quad \forall^1 \quad \longrightarrow}{\forall \qquad\qquad\qquad \longrightarrow}$$

The Curry-Howard Isomorphism

$$\frac{\Pi \quad \rightarrow \quad \times \quad + \quad \perp}{\forall \quad \supset \quad \wedge \quad \vee \quad F}$$

The Girard-Reynolds Isomorphism

$$\frac{\forall \quad \forall^2 \quad \forall^1 \quad \rightarrow}{\forall \qquad\qquad\qquad \rightarrow}$$

Rather than enriching the type systems to match logic,
we impoverish logic to match the type structure.

— Daniel Leivant

Part I

Overview

The Girard projection

$$\mathbf{N} \;\equiv\; \{n^{\mathsf{N}} \mid \forall \boldsymbol{\mathcal{X}}^{\mathsf{N}}.\, (\forall m^{\mathsf{N}}.\, m \in \boldsymbol{\mathcal{X}} \to \mathsf{s}\, m \in \boldsymbol{\mathcal{X}}) \to \mathsf{z} \in \boldsymbol{\mathcal{X}} \to n \in \boldsymbol{\mathcal{X}}\}$$

$$\mathbf{N}^{\circ} \;\equiv\; \forall X.\, (X \to X) \to (X \to X)$$

The Reynolds embedding

$$\mathsf{N} \;\equiv\; \forall X.\, (X \to X) \to (X \to X)$$

$$\mathsf{N}^{*} \;\equiv\; \{n^{\mathsf{N}} \mid \forall X.\, \forall \boldsymbol{\mathcal{X}}^{X}.\, \forall s^{X \to X}.\, (\forall m^{X}.\, m \in \boldsymbol{\mathcal{X}} \to s\, m \in \boldsymbol{\mathcal{X}}) \to \forall z^{X}.\, z \in \boldsymbol{\mathcal{X}} \to n\; X\; s\; z \in \boldsymbol{\mathcal{X}}\}$$

Doubling and Parametricity

$$\mathsf{N} \equiv \forall X.\,(X \to X) \to (X \to X)$$

$$\mathsf{N}^* \equiv$$
$$\{n^{\mathsf{N}} \mid$$
$$\forall X.\,\forall \mathcal{X}^X.$$
$$\forall s^{X \to X}.\,(\forall m^X.\,m \in \mathcal{X} \to s\;m \in \mathcal{X}) \to$$
$$\forall z^X.\,z \in \mathcal{X} \to n\;X\;s\;z \in \mathcal{X}\}$$

$$\mathsf{N}^{*\ddagger} \equiv$$
$$\{(n^{\mathsf{N}}, n'^{\mathsf{N}}) \mid$$
$$\forall X.\,\forall X'.\,\forall \mathcal{X}^{X \times X'}.$$
$$\forall s^{X \to X}.\,\forall s'^{X' \to X'}.\,(\forall m^X.\,\forall m'^{X'}.\,(m, m') \in \mathcal{X} \to (s\;m, s'\;m') \in \mathcal{X}) \to$$
$$\forall z^X.\,\forall z'^{X'}.\,(z, z') \in \mathcal{X} \to (n\;X\;s\;z, n'\;X'\;s'\;z') \in \mathcal{X}\}$$

Part II

Details

Second-order lambda calculus (F2)

$$\dfrac{\begin{array}{c}[x^A]\\ \vdots \\ u^B\end{array}}{(\lambda x^A.\, u)^{A \to B}} \to\text{-I}^x \qquad \dfrac{s^{A \to B} \qquad t^A}{(s\; t)^B} \to\text{-E}$$

$$\dfrac{u^B}{(\Lambda X.\, u)^{\forall X.\, B}} \forall\text{-I} \quad X \text{ does not escape} \qquad \dfrac{s^{\forall X.\, B}}{(s\; A)^{B[A/X]}} \forall\text{-E}$$

Second-order propositional logic (P2)

$$\frac{\begin{array}{c}[\boldsymbol{A}]^{\boldsymbol{x}} \\ \vdots \\ \boldsymbol{B}\end{array}}{\boldsymbol{A} \to \boldsymbol{B}} \to\text{-}\mathbf{I}^{\boldsymbol{x}} \qquad \frac{\boldsymbol{A} \to \boldsymbol{B} \qquad \boldsymbol{A}}{\boldsymbol{B}} \to\text{-}\mathbf{E}$$

$$\frac{\boldsymbol{B}}{\forall \boldsymbol{\mathcal{X}}^{C}.\, \boldsymbol{B}} \forall\text{-}\mathbf{I} \qquad \boldsymbol{\mathcal{X}} \text{ does not escape} \qquad \frac{\forall \boldsymbol{\mathcal{X}}^{C}.\, \boldsymbol{B}}{\boldsymbol{B}[\boldsymbol{\mathcal{A}}^{C}/\boldsymbol{\mathcal{X}}]} \forall\text{-}\mathbf{E}$$

$$\frac{\boldsymbol{B}}{\forall x^{C}.\, \boldsymbol{B}} \forall^{1}\text{-I} \qquad x \text{ does not escape} \qquad \frac{\forall x^{C}.\, \boldsymbol{B}}{\boldsymbol{B}[t^{C}/x]} \forall^{1}\text{-E}$$

$$\frac{\boldsymbol{B}}{\forall X.\, \boldsymbol{B}} \forall^{2}\text{-I} \qquad X \text{ does not escape} \qquad \frac{\forall X.\, \boldsymbol{B}}{\boldsymbol{B}[A/X]} \forall^{2}\text{-E}$$

β rules

$$
\begin{aligned}
(\lambda x^T.\, u)\ t &\quad =_\beta \quad u[t/x] \\
(\Lambda X.\, u)\ A &\quad =_\beta \quad u[A/X] \\
t^C \in \{x^C \mid \boldsymbol{A}\} &\quad =_\beta \quad \boldsymbol{A}[t/x]
\end{aligned}
$$

$$
\frac{\boldsymbol{A}}{\boldsymbol{B}}\,\beta \qquad \boldsymbol{A} =_\beta \boldsymbol{B}
$$

Part III

Girard projection

Girard projection

Propositions

$$
\begin{aligned}
(t^C \in \boldsymbol{\mathcal{A}}^C)^\circ &\equiv \boldsymbol{\mathcal{A}}^\circ \\
(\boldsymbol{A} \to \boldsymbol{B})^\circ &\equiv \boldsymbol{A}^\circ \to \boldsymbol{B}^\circ \\
(\forall \boldsymbol{\mathcal{X}}^C.\, \boldsymbol{B})^\circ &\equiv \forall X.\, \boldsymbol{B}^\circ \\
(\forall x^C.\, \boldsymbol{B})^\circ &\equiv \boldsymbol{B}^\circ \\
(\forall X.\, \boldsymbol{B})^\circ &\equiv \boldsymbol{B}^\circ
\end{aligned}
$$

Predicates

$$
\begin{aligned}
(\boldsymbol{\mathcal{X}}^C)^\circ &\equiv X \\
(\{x^C \mid \boldsymbol{A}\})^\circ &\equiv \boldsymbol{A}^\circ
\end{aligned}
$$

Girard projection

$$\left(\dfrac{\begin{array}{c}[\boldsymbol{A}]^{\boldsymbol{x}}\\ \vdots\; \boldsymbol{u}\\ \boldsymbol{B}\end{array}}{\boldsymbol{A} \to \boldsymbol{B}}\;\boldsymbol{\to}\textbf{-I}^{\boldsymbol{x}}\right)^{\circ} \equiv \dfrac{\begin{array}{c}[x^{\boldsymbol{A}^\circ}]\\ \vdots\\ \boldsymbol{u}^{\circ\,\boldsymbol{B}^\circ}\end{array}}{(\lambda x^{\boldsymbol{A}^\circ}.\, \boldsymbol{u}^\circ)^{\boldsymbol{A}^\circ\to \boldsymbol{B}^\circ}}\;\to\text{-I}^{x}$$

$$\left(\dfrac{\begin{array}{c}\vdots\; \boldsymbol{s}\\ \boldsymbol{A}\to\boldsymbol{B}\end{array}\qquad\begin{array}{c}\vdots\; \boldsymbol{t}\\ \boldsymbol{A}\end{array}}{\boldsymbol{B}}\;\boldsymbol{\to}\textbf{-E}\right)^{\circ} \equiv \dfrac{\begin{array}{c}\vdots\\ \boldsymbol{s}^{\circ\,\boldsymbol{A}^\circ\to\boldsymbol{B}^\circ}\end{array}\qquad\begin{array}{c}\vdots\\ \boldsymbol{t}^{\circ\,\boldsymbol{A}^\circ}\end{array}}{(\boldsymbol{s}^\circ\; \boldsymbol{t}^\circ)^{\boldsymbol{B}^\circ}}\;\to\text{-E}$$

Girard projection

$$\left(\dfrac{\overset{\vdots\; u}{B}}{\forall \mathcal{X}^{C}.\, B}\;\forall\text{-I}\right)^{\circ} \equiv \dfrac{\overset{\vdots}{u^{\circ\, B^{\circ}}}}{(\Lambda X.\, u^{\circ})^{\forall X.\, B^{\circ}}}\;\forall\text{-I}$$

$$\left(\dfrac{\overset{\vdots\; s}{\forall \mathcal{X}^{C}.\, B}}{B[\mathcal{A}^{C}/\mathcal{X}]}\;\forall\text{-E}\right)^{\circ} \equiv \dfrac{\overset{\vdots}{s^{\circ\, \forall X.\, B^{\circ}}}}{(s^{\circ}\; \mathcal{A}^{\circ})^{B^{\circ}[\mathcal{A}^{\circ}/X]}}\;\forall\text{-E}$$

Girard projection

$$\left(\dfrac{\begin{array}{c}\vdots\ \boldsymbol{u}\\ \boldsymbol{B}\end{array}}{\forall x^{C}.\,\boldsymbol{B}}\ \forall^{1}\text{-I}\right)^{\circ} \equiv \begin{array}{c}\vdots\\ \boldsymbol{u}^{\circ}\,\boldsymbol{B}^{\circ}\end{array} \qquad \left(\dfrac{\begin{array}{c}\vdots\ \boldsymbol{s}\\ \forall x^{C}.\,\boldsymbol{B}\end{array}}{\boldsymbol{B}[t^{C}/x]}\ \forall^{1}\text{-E}\right)^{\circ} \equiv \begin{array}{c}\vdots\\ \boldsymbol{s}^{\circ}\,\boldsymbol{B}^{\circ}\end{array}$$

$$\left(\dfrac{\begin{array}{c}\vdots\ \boldsymbol{u}\\ \boldsymbol{B}\end{array}}{\forall X.\,\boldsymbol{B}}\ \forall^{2}\text{-I}\right)^{\circ} \equiv \begin{array}{c}\vdots\\ \boldsymbol{u}^{\circ}\,\boldsymbol{B}^{\circ}\end{array} \qquad \left(\dfrac{\begin{array}{c}\vdots\ \boldsymbol{s}\\ \forall X.\,\boldsymbol{B}\end{array}}{\boldsymbol{B}[A/X]}\ \forall^{2}\text{-E}\right)^{\circ} \equiv \begin{array}{c}\vdots\\ \boldsymbol{s}^{\circ}\,\boldsymbol{B}^{\circ}\end{array}$$

$$\left(\dfrac{\begin{array}{c}\vdots\ \boldsymbol{t}\\ \boldsymbol{A}\end{array}}{\boldsymbol{B}}\ \beta\right)^{\circ} \equiv \begin{array}{c}\vdots\\ \boldsymbol{t}^{\circ}\,\boldsymbol{A}^{\circ}\end{array}$$

Part IV

Reynolds embedding

Reynolds embedding

Types

$$(X)^* \equiv \boldsymbol{\mathcal{X}}^X$$

$$(A \to B)^* \equiv \{z^{A\to B} \mid \forall x^A.\, x \in A^* \to z\,x \in B^*\}$$

$$(\forall X.\, B)^* \equiv \{z^{\forall X.\, B} \mid \forall X.\, \forall \boldsymbol{\mathcal{X}}^X.\, z\,X \in B^*\}$$

Reynolds embedding

$$\left(\dfrac{\begin{array}{c}[x^A]\\ \vdots\\ u^B\end{array}}{(\lambda x^A.\,u)^{A\to B}}\to\text{-I}^x\right)^* \equiv \dfrac{\dfrac{\dfrac{\begin{array}{c}[x\in A^*]^{x}\\ \vdots\ u^*\\ u\in B^*\end{array}}{(\lambda x^A.\,u)\,x\in B^*}\,\beta}{x\in A^*\to(\lambda x^A.\,u)\,x\in B^*}\,\to\text{-I}^{x}}{\forall x^A.\,x\in A^*\to(\lambda x^A.\,u)\,x\in B^*}\,\forall^1\text{-I}$$

$$\left(\dfrac{\begin{array}{c}\vdots\\ s^{A\to B}\end{array}\qquad\begin{array}{c}\vdots\\ t^A\end{array}}{(s\,t)^B}\to\text{-E}\right)^* \equiv \dfrac{\dfrac{\begin{array}{c}\vdots\ s^*\\ \forall x^A.\,x\in A^*\to s\,x\in B^*\end{array}}{t\in A^*\to s\,t\in B^*}\,\forall^1\text{-E}\qquad \begin{array}{c}\vdots\ t^*\\ t\in A^*\end{array}}{s\,t\in B^*}\,\to\text{-E}$$

Reynolds embedding

$$\left(\dfrac{\begin{array}{c}\vdots\\ u^{B}\end{array}}{(\Lambda X.\, u)^{\forall X.\, B}}\ \forall\text{-I}\right)^{*} \equiv \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ u^{*}\\ u \in B^{*}\end{array}}{(\Lambda X.\, u)\ X \in B^{*}}\ \beta}{\forall \boldsymbol{\mathcal{X}}^{X}\boldsymbol{.}\ (\Lambda X.\, u)\ X \in B^{*}}\ \boldsymbol{\forall\text{-I}}}{\forall X.\, \forall \boldsymbol{\mathcal{X}}^{X}\boldsymbol{.}\ (\Lambda X.\, u)\ X \in B^{*}}\ \forall^{2}\text{-I}$$

$$\left(\dfrac{\begin{array}{c}\vdots\\ s^{\forall X.\, B}\end{array}}{(s\ A)^{B[A/X]}}\ \forall\text{-E}\right)^{*} \equiv \dfrac{\dfrac{\begin{array}{c}\vdots\ s^{*}\\ \forall X.\, \forall \boldsymbol{\mathcal{X}}^{X}\boldsymbol{.}\ s\ X \in B^{*}\end{array}}{\forall \boldsymbol{\mathcal{X}}^{A}\boldsymbol{.}\ s\ A \in B^{*}[A/X]}\ \forall^{2}\text{-E}}{s\ A \in B^{*}[A/X, A^{*}/\boldsymbol{\mathcal{X}}]}\ \boldsymbol{\forall\text{-E}}$$

Part V

Doubling

Doubling

Propositions

$$(t^C \in \boldsymbol{\mathcal{A}}^C)^\ddagger \equiv (t^C, t'^{C'}) \in \boldsymbol{\mathcal{A}}^{\ddagger\, C \times C'}$$

$$(\boldsymbol{A} \to \boldsymbol{B})^\ddagger \equiv \boldsymbol{A}^\ddagger \to \boldsymbol{B}^\ddagger$$

$$(\forall \boldsymbol{\mathcal{X}}^C.\, \boldsymbol{B})^\ddagger \equiv \forall \boldsymbol{\mathcal{X}}^{C \times C'}.\, \boldsymbol{B}^\ddagger$$

$$(\forall x^C.\, \boldsymbol{B})^\ddagger \equiv \forall x^C, x'^{C'}.\, \boldsymbol{B}^\ddagger$$

$$(\forall X.\, \boldsymbol{B})^\ddagger \equiv \forall X, X'.\, \boldsymbol{B}^\ddagger$$

Predicates

$$(\boldsymbol{\mathcal{X}}^C)^\ddagger \equiv \boldsymbol{\mathcal{X}}^{C \times C'}$$

$$(\{x^C \mid \boldsymbol{A}\})^\ddagger \equiv \{(x^C, x'^{C'}) \mid \boldsymbol{A}^\ddagger\}$$

Doubling

$$\left(\dfrac{\begin{array}{c}[A]^{x}\\ \vdots\ u\\ B\end{array}}{A \longrightarrow B}\longrightarrow\text{-I}^{x}\right)^{\ddagger} \equiv \dfrac{\begin{array}{c}[A^{\ddagger}]^{x}\\ \vdots\ u^{\ddagger}\\ B^{\ddagger}\end{array}}{A^{\ddagger} \longrightarrow B^{\ddagger}}\longrightarrow\text{-I}^{x}$$

$$\left(\dfrac{\begin{array}{cc}\vdots\ s & \vdots\ t\\ A \longrightarrow B & A\end{array}}{B}\longrightarrow\text{-E}\right)^{\ddagger} \equiv \dfrac{\begin{array}{cc}\vdots\ s^{\ddagger} & \vdots\ t^{\ddagger}\\ A^{\ddagger} \longrightarrow B^{\ddagger} & A^{\ddagger}\end{array}}{B^{\ddagger}}\longrightarrow\text{-E}$$

Doubling

$$\left(\dfrac{\begin{array}{c}\vdots\; \boldsymbol{u} \\ \boldsymbol{B}\end{array}}{\forall \boldsymbol{\mathcal{X}}^{C}.\, \boldsymbol{B}}\ \forall\text{-}\mathbf{I}\right)^{\ddagger} \equiv \dfrac{\begin{array}{c}\vdots\; \boldsymbol{u}^{\ddagger} \\ \boldsymbol{B}^{\ddagger}\end{array}}{\forall \boldsymbol{\mathcal{X}}^{C\times C'}.\, \boldsymbol{B}^{\ddagger}}\ \forall\text{-}\mathbf{I}$$

$$\left(\dfrac{\begin{array}{c}\vdots\; \boldsymbol{s} \\ \forall \boldsymbol{\mathcal{X}}^{C}.\, \boldsymbol{B}\end{array}}{\boldsymbol{B}[\boldsymbol{\mathcal{A}}^{C}/\boldsymbol{\mathcal{X}}]}\ \forall\text{-}\mathbf{E}\right)^{\ddagger} \equiv \dfrac{\begin{array}{c}\vdots\; \boldsymbol{s}^{\ddagger} \\ \forall \boldsymbol{\mathcal{X}}^{C\times C'}.\, \boldsymbol{B}^{\ddagger}\end{array}}{\boldsymbol{B}^{\ddagger}[\boldsymbol{\mathcal{A}}^{\ddagger\, C\times C'}/\boldsymbol{\mathcal{X}}]}\ \forall\text{-}\mathbf{E}$$

Doubling

$$\left(\dfrac{\begin{array}{c}\vdots\ \boldsymbol{u} \\ \boldsymbol{B}\end{array}}{\forall x^{C}.\,\boldsymbol{B}}\ \forall^1\text{-I} \right)^{\ddagger} \equiv \dfrac{\begin{array}{c}\vdots\ \boldsymbol{u}^{\ddagger} \\ \boldsymbol{B}^{\ddagger}\end{array}}{\forall x^{C}, x'^{C'}.\,\boldsymbol{B}^{\ddagger}}\ \forall^1\text{-I twice}$$

$$\left(\dfrac{\begin{array}{c}\vdots\ \boldsymbol{s} \\ \forall x^{C}.\,\boldsymbol{B}\end{array}}{\boldsymbol{B}[t^{C}/x]}\ \forall^1\text{-E} \right)^{\ddagger} \equiv \dfrac{\begin{array}{c}\vdots\ \boldsymbol{s}^{\ddagger} \\ \forall x^{C}, x'^{C'}.\,\boldsymbol{B}^{\ddagger}\end{array}}{\boldsymbol{B}^{\ddagger}[t^{C}/x, t'^{C'}/x']}\ \forall^1\text{-E twice}$$

Doubling

$$\left(\dfrac{\begin{array}{c}\vdots\, \boldsymbol{u}\\ \boldsymbol{B}\end{array}}{\forall X.\, \boldsymbol{B}}\,\forall^2\text{-I}\right)^{\ddagger} \equiv \dfrac{\begin{array}{c}\vdots\, \boldsymbol{u}^{\ddagger}\\ \boldsymbol{B}^{\ddagger}\end{array}}{\forall X, X'.\, \boldsymbol{B}^{\ddagger}}\,\forall^2\text{-I twice}$$

$$\left(\dfrac{\begin{array}{c}\vdots\, \boldsymbol{s}\\ \forall X.\, \boldsymbol{B}\end{array}}{\boldsymbol{B}[A/X]}\,\forall^2\text{-E}\right)^{\ddagger} \equiv \dfrac{\begin{array}{c}\vdots\, \boldsymbol{s}^{\ddagger}\\ \forall X, X'.\, \boldsymbol{B}^{\ddagger}\end{array}}{\boldsymbol{B}^{\ddagger}[A/X, A'/X']}\,\forall^2\text{-E twice}$$

Doubling

$$\left(\dfrac{\begin{array}{c}\vdots\ \boldsymbol{t} \\ \boldsymbol{A}\end{array}}{\boldsymbol{B}}\,\beta \right)^{\ddagger} \equiv \dfrac{\begin{array}{c}\vdots\ \boldsymbol{t}^{\ddagger} \\ \boldsymbol{A}^{\ddagger}\end{array}}{\boldsymbol{B}^{\ddagger}}\,\beta \text{ twice}$$

Part VI

Induction and Parametricity

Induction and Parametricity

Proposition.

$$\forall n.\, n \in \mathbf{N} \longrightarrow n \in \mathsf{N}^*$$

Proposition.

$$\forall n, n'.\, (n, n') \in \mathsf{N}^{*\ddagger} \leftrightarrow n = n' \wedge n \in \mathbf{N}$$

Corollary.

$$(\forall n.\, n \in \mathsf{N}^* \longrightarrow (n, n) \in \mathsf{N}^{*\ddagger}) \leftrightarrow (\forall n.\, n \in \mathsf{N}^* \longrightarrow n \in \mathbf{N})$$

Part VII

Successor

$$\boldsymbol{A}_{\mathsf{s}} \equiv \forall m^{\mathsf{N}}.\, m \in \boldsymbol{\mathcal{X}} \longrightarrow \mathsf{s}\, m \in \boldsymbol{\mathcal{X}} \qquad\qquad \boldsymbol{A}_{\mathsf{z}} \equiv \mathsf{z} \in \boldsymbol{\mathcal{X}}$$

$$\dfrac{[n \in \mathbf{N}]^{\boldsymbol{n}}}{\forall \boldsymbol{\mathcal{X}}^{\mathsf{N}}.\, \boldsymbol{A}_{\mathsf{s}} \longrightarrow \boldsymbol{A}_{\mathsf{z}} \longrightarrow n \in \boldsymbol{\mathcal{X}}}\ \beta$$

$$\dfrac{\forall \boldsymbol{\mathcal{X}}^{\mathsf{N}}.\, \boldsymbol{A}_{\mathsf{s}} \longrightarrow \boldsymbol{A}_{\mathsf{z}} \longrightarrow n \in \boldsymbol{\mathcal{X}}}{\boldsymbol{A}_{\mathsf{s}} \longrightarrow \boldsymbol{A}_{\mathsf{z}} \longrightarrow n \in \boldsymbol{\mathcal{X}}}\ \forall\text{-}\mathbf{E}$$

$$\dfrac{\boldsymbol{A}_{\mathsf{s}} \longrightarrow \boldsymbol{A}_{\mathsf{z}} \longrightarrow n \in \boldsymbol{\mathcal{X}} \qquad [\boldsymbol{A}_{\mathsf{s}}]^{\boldsymbol{s}}}{\boldsymbol{A}_{\mathsf{z}} \longrightarrow n \in \boldsymbol{\mathcal{X}}}\ \longrightarrow\text{-}\mathbf{E}$$

$$\dfrac{\boldsymbol{A}_{\mathsf{z}} \longrightarrow n \in \boldsymbol{\mathcal{X}} \qquad [\boldsymbol{A}_{\mathsf{z}}]^{\boldsymbol{z}}}{n \in \boldsymbol{\mathcal{X}}}\ \longrightarrow\text{-}\mathbf{E}$$

$$\dfrac{[\boldsymbol{A}_{\mathsf{s}}]^{\boldsymbol{s}}}{n \in \boldsymbol{\mathcal{X}} \longrightarrow \mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}\ \forall^1\text{-E}$$

$$\dfrac{n \in \boldsymbol{\mathcal{X}} \longrightarrow \mathsf{s}\, n \in \boldsymbol{\mathcal{X}} \qquad n \in \boldsymbol{\mathcal{X}}}{\mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}\ \longrightarrow\text{-}\mathbf{E}$$

$$\dfrac{\mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}{\boldsymbol{A}_{\mathsf{z}} \longrightarrow \mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}\ \longrightarrow\text{-}\mathbf{I}^{\boldsymbol{z}}$$

$$\dfrac{\boldsymbol{A}_{\mathsf{z}} \longrightarrow \mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}{\boldsymbol{A}_{\mathsf{s}} \longrightarrow \boldsymbol{A}_{\mathsf{z}} \longrightarrow \mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}\ \longrightarrow\text{-}\mathbf{I}^{\boldsymbol{s}}$$

$$\dfrac{\boldsymbol{A}_{\mathsf{s}} \longrightarrow \boldsymbol{A}_{\mathsf{z}} \longrightarrow \mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}{\forall \boldsymbol{\mathcal{X}}^{\mathsf{N}}.\, \boldsymbol{A}_{\mathsf{s}} \longrightarrow \boldsymbol{A}_{\mathsf{z}} \longrightarrow \mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}\ \forall\text{-}\mathbf{I}$$

$$\dfrac{\forall \boldsymbol{\mathcal{X}}^{\mathsf{N}}.\, \boldsymbol{A}_{\mathsf{s}} \longrightarrow \boldsymbol{A}_{\mathsf{z}} \longrightarrow \mathsf{s}\, n \in \boldsymbol{\mathcal{X}}}{\mathsf{s}\, n \in \mathbf{N}}\ \beta$$

$$\dfrac{\mathsf{s}\, n \in \mathbf{N}}{n \in \mathbf{N} \longrightarrow \mathsf{s}\, n \in \mathbf{N}}\ \longrightarrow\text{-}\mathbf{I}^{\boldsymbol{n}}$$

$$\dfrac{n \in \mathbf{N} \longrightarrow \mathsf{s}\, n \in \mathbf{N}}{\forall n^{\mathsf{N}}.\, n \in \mathbf{N} \longrightarrow \mathsf{s}\, n \in \mathbf{N}}\ \forall^1\text{-I}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[s^{X \to X}] \qquad \dfrac{\dfrac{\dfrac{[n^{\mathsf{N}}]}{(n\ X)^{(X \to X) \to X \to X}}\ \forall\text{-E} \qquad [s^{X \to X}]}{(n\ X\ s)^{X \to X}}\ {\to}\text{-E} \qquad [z^{X}]}{(n\ X\ s\ z)^{X}}\ {\to}\text{-E}}{(s\ (n\ X\ s\ z))^{X}}\ {\to}\text{-E}}{(\lambda z^{X}.\, s\ (n\ X\ s\ z))^{X \to X}}\ {\to}\text{-I}^{z}}{(\lambda s^{X \to X}.\, \lambda z^{X}.\, s\ (n\ X\ s\ z))^{(X \to X) \to X \to X}}\ {\to}\text{-I}^{s}}{(\Lambda X.\, \lambda s^{X \to X}.\, \lambda z^{X}.\, s\ (n\ X\ s\ z))^{\mathsf{N}}}\ \forall\text{-I}}{(\lambda n^{\mathsf{N}}.\, \Lambda X.\, \lambda s^{X \to X}.\, \lambda z^{X}.\, s\ (n\ X\ s\ z))^{\mathsf{N} \to \mathsf{N}}}\ {\to}\text{-I}^{n}
$$

Part VIII

Conclusion

Related work

Girard 1972

Reynolds 1974, 1983

Böhm and Beararducci 1985

Leivant 1990

Krivine and Parigot 1990

Mairson 1991

Hasegawa 1994

Plotkin and Abadi 1993

Takeuti 1998

Related work: Models

Moggi 1986

Breazu-Tannen and Coquand 1988

Freyd 1989

Hyland, Robinson, and Rosolini 1990

Rummelhoff 2003

Møgelberg 2004