

# Call-by-name is dual to call-by-value

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Part 1

A Deal with the Devil

Either

(a) I will give you \$1,000,000,000

or

(b) I will grant you one wish if you pay me  
\$1,000,000,000.

Either

(a) I will give you \$1,000,000,000

or

(b) I will grant you one wish if you pay me  
\$1,000,000,000.

Either

(a) I will give you \$1,000,000,000

or

(b) I will grant you one wish if you pay me  
\$1,000,000,000.

Part 2

A Question

$$\frac{\text{Lambda Calculus}}{\text{Natural Deduction}} = \frac{?}{\text{Sequent Calculus}}$$

(Intuitionistic) (Classical)

$$\frac{\text{Lambda Calculus}}{\text{Natural Deduction}} = \frac{\text{Dual Calculus}}{\text{Sequent Calculus}}$$

(Intuitionistic) (Classical)



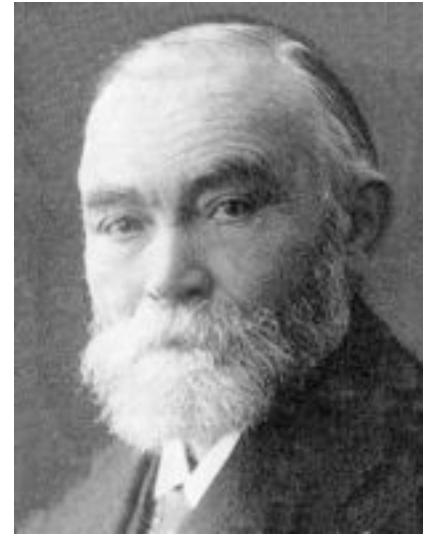
Part 3

The History

# Logic



George Boole (1815–1864)



Gottlob Frege (1848–1925)

Boole (1847): *Laws of Thought*

Frege (1879): *Begriffsschrift*

# Duality

$$A \& \neg A = \perp$$

$$A \vee \neg A = \top$$

$$A \& (B \vee C) = (A \& B) \vee (A \& C)$$

$$A \vee (B \& C) = (A \vee B) \& (A \vee C)$$

Poncelet (1818), Gergonne (1826): duals in projective geometry

Boole (1847), Frege (1879): no duality!

Schröder (1890): duals in logic

# Natural Deduction and Lambda Calculus



Alonzo Church (1903–1995)



Gerhard Gentzen (1909–1945)

Church (1932):  $\lambda$ -calculus

Gentzen (1935): natural deduction

Church (1940): simply-typed  $\lambda$ -calculus

# The Curry-Howard Isomorphism



Haskell Curry (1900–1982)



William Howard

Curry and Feys (1958): combinatory logic

Prawitz (1965): proof reduction

de Bruijn (1968): encoding of proofs

Howard (1980): natural deduction  $\simeq$   $\lambda$ -calculus

# Curry-Howard for Classical Logic

Gentzen (1935): sequent calculus

Filinski (1989): symmetric  $\lambda$ -calculus

Griffin (1990): Curry-Howard for classical logic

Parigot (1992):  $\lambda\mu$ -calculus

Danos, Joinet, and Schellinx (1995): dual encodings in linear logic

Barbanera and Berardi (1996): symmetric  $\lambda$ -calculus

Streicher and Reus (1998): dual cps transforms

Selinger (1998): dual control categories

Curien and Herbelin (2000): Curry-Howard for classical sequent calculus

Wadler (2003): dual calculus

## Part 4

# Natural Deduction (Intuitionistic)

# Gentzen 1935: Natural Deduction

$\&-I$ $\frac{\mathcal{A} \quad \mathcal{B}}{\mathcal{A} \& \mathcal{B}}$	$\&-E$ $\frac{\mathcal{A} \& \mathcal{B}}{\mathcal{A}} \quad \frac{\mathcal{A} \& \mathcal{B}}{\mathcal{B}}$	$\vee-I$ $\frac{\mathcal{A}}{\mathcal{A} \vee \mathcal{B}} \quad \frac{\mathcal{B}}{\mathcal{A} \vee \mathcal{B}}$	$\vee-E$ $\frac{\mathcal{A} \vee \mathcal{B} \quad \begin{array}{l} [\mathcal{A}] \\ \mathcal{C} \end{array} \quad \begin{array}{l} [\mathcal{B}] \\ \mathcal{C} \end{array}}{\mathcal{C}}$
$\forall-I$ $\frac{\mathcal{F}a}{\forall x \mathcal{F}x}$	$\forall-E$ $\frac{\forall x \mathcal{F}x}{\mathcal{F}a}$	$\exists-I$ $\frac{\mathcal{F}a}{\exists x \mathcal{F}x}$	$\exists-E$ $\frac{\exists x \mathcal{F}x \quad \begin{array}{l} [\mathcal{F}a] \\ \mathcal{C} \end{array}}{\mathcal{C}}$
$\supset-I$ $\frac{\begin{array}{l} [\mathcal{A}] \\ \mathcal{B} \end{array}}{\mathcal{A} \supset \mathcal{B}}$	$\supset-E$ $\frac{\mathcal{A} \quad \mathcal{A} \supset \mathcal{B}}{\mathcal{B}}$	$\neg-I$ $\frac{\begin{array}{l} [\mathcal{A}] \\ \wedge \end{array}}{\neg \mathcal{A}}$	$\neg-E$ $\frac{\mathcal{A} \quad \neg \mathcal{A}}{\wedge} \quad \frac{\wedge}{\mathcal{D}}$



# Gentzen 1935: Natural Deduction

$$\frac{\begin{array}{c} \vdots \\ A \end{array} \quad \begin{array}{c} \vdots \\ B \end{array}}{A \& B} \&I \quad \frac{\begin{array}{c} \vdots \\ A \& B \end{array}}{A} \quad \frac{\begin{array}{c} \vdots \\ A \& B \end{array}}{B} \&E$$

$$\frac{\begin{array}{c} \vdots \\ A \end{array}}{A \vee B} \quad \frac{\begin{array}{c} \vdots \\ B \end{array}}{A \vee B} \vee I \quad \frac{\begin{array}{c} \vdots \\ A \vee B \end{array} \quad \begin{array}{c} \{A\} \\ \vdots \\ C \end{array} \quad \begin{array}{c} \{B\} \\ \vdots \\ C \end{array}}{C} \vee E$$

$$\frac{\begin{array}{c} \{A\} \\ \vdots \\ B \end{array}}{A \supset B} \supset I \quad \frac{\begin{array}{c} \vdots \\ A \supset B \end{array} \quad \begin{array}{c} \vdots \\ A \end{array}}{B} \supset E$$

# Prawitz 1965: Simplifying proofs

$$\frac{\frac{\begin{array}{c} \vdots \\ A \end{array} \quad \begin{array}{c} \vdots \\ B \end{array}}{A \& B} \&I}{A} \&E \longrightarrow \begin{array}{c} \vdots \\ A \end{array}$$

$$\frac{\frac{\begin{array}{c} \vdots \\ A \end{array}}{A \vee B} \vee I \quad \begin{array}{c} \{A\} \\ \vdots \\ C \end{array} \quad \begin{array}{c} \{B\} \\ \vdots \\ C \end{array}}{C} \vee E}{C} \vee E \longrightarrow \begin{array}{c} \vdots \\ A \\ \vdots \\ C \end{array}$$

$$\frac{\frac{\begin{array}{c} \{A\} \\ \vdots \\ B \end{array}}{A \supset B} \supset I \quad \begin{array}{c} \vdots \\ A \end{array}}{B} \supset E}{B} \supset E \longrightarrow \begin{array}{c} \vdots \\ A \\ \vdots \\ B \end{array}$$

Part 5

The Lambda Calculus

# Church 1932: Lambda Calculus

An occurrence of a variable  $\mathbf{x}$  in a given formula is called an occurrence of  $\mathbf{x}$  as a *bound variable* in the given formula if it is an occurrence of  $\mathbf{x}$  in a part of the formula of the form  $\lambda \mathbf{x}[\mathbf{M}]$ ; that is, if there is a formula  $\mathbf{M}$  such that  $\lambda \mathbf{x}[\mathbf{M}]$  occurs in the given formula and the occurrence of  $\mathbf{x}$  in question is an occurrence in  $\lambda \mathbf{x}[\mathbf{M}]$ . All other occurrences of a variable in a formula are called occurrences as a *free variable*.

A formula is said to be *well-formed* if it is a variable, or if it is one

# Church 1940: Simply-typed Lambda Calculus

$$\frac{\begin{array}{c} \vdots \\ M : A \end{array} \quad \begin{array}{c} \vdots \\ N : B \end{array}}{(M, N) : A \& B} \&I$$

$$\frac{\begin{array}{c} \vdots \\ O : A \& B \end{array}}{\text{fst } O : A} \quad \frac{\begin{array}{c} \vdots \\ O : A \& B \end{array}}{\text{snd } O : B} \&E$$

$$\frac{\begin{array}{c} \vdots \\ M : A \end{array}}{\text{inl } M : A \vee B} \quad \frac{\begin{array}{c} \vdots \\ N : B \end{array}}{\text{inr } N : A \vee B} \vee I$$

$$\frac{\begin{array}{c} \vdots \\ O : A \vee B \end{array} \quad \begin{array}{c} \{x : A\} \\ \vdots \\ P : C \end{array} \quad \begin{array}{c} \{y : B\} \\ \vdots \\ Q : C \end{array}}{\text{case } O \text{ of inl } x \Rightarrow P, \text{ inr } y \Rightarrow Q : C} \vee E$$

$$\frac{\begin{array}{c} \{x : A\} \\ \vdots \\ N : B \end{array}}{\lambda x. N : A \supset B} \supset I$$

$$\frac{\begin{array}{c} \vdots \\ O : A \supset B \end{array} \quad \begin{array}{c} \vdots \\ M : A \end{array}}{O M : B} \supset E$$

# Church 1932, 1940: Reducing terms

$$\frac{\frac{\frac{\vdots}{M:A} \quad \frac{\vdots}{N:B}}{(M, N):A \& B} \&I}{\text{fst}(M, N):A} \&E \longrightarrow \frac{\vdots}{M:A}$$

$$\frac{\frac{\frac{\vdots}{M:A} \quad \frac{\{x:A\} \quad \{y:B\}}{P:C} \quad \frac{\{y:B\}}{Q:C}}{\text{inl } M:A \vee B} \vee I}{\text{case (inl } M) \text{ of inl } x \Rightarrow P, \text{ inr } y \Rightarrow Q:C} \vee E \longrightarrow \frac{\frac{\vdots}{M:A} \quad \frac{\vdots}{P\{M/x\}:C}}{P\{M/x\}:C}$$

$$\frac{\frac{\frac{\{x:A\} \quad \frac{\vdots}{N:B}}{\lambda x. N:A \supset B} \supset I}{(\lambda x. N)M:B} \supset E \quad \frac{\vdots}{M:A}}{N\{M/x\}:B} \longrightarrow \frac{\frac{\{x:A\} \quad \frac{\vdots}{N:B}}{\lambda x. N:A \supset B} \supset I \quad \frac{\vdots}{M:A}}{(\lambda x. N)M:B} \supset E \longrightarrow \frac{\frac{\vdots}{M:A} \quad \frac{\vdots}{N\{M/x\}:B}}{N\{M/x\}:B}$$

# Church 1932: Call-by-name

$(\beta\&)$	$\text{fst } (M, N)$	$\longrightarrow_n$	$M$
$(\beta\&)$	$\text{snd } (M, N)$	$\longrightarrow_n$	$N$
$(\beta\vee)$	$\text{case } (\text{inl } M) \text{ of } \text{inl } x \Rightarrow P, \text{inr } y \Rightarrow Q$	$\longrightarrow_n$	$P\{M/x\}$
$(\beta\vee)$	$\text{case } (\text{inr } N) \text{ of } \text{inl } x \Rightarrow P, \text{inr } y \Rightarrow Q$	$\longrightarrow_n$	$Q\{N/y\}$
$(\beta\supset)$	$(\lambda x. N) M$	$\longrightarrow_n$	$N\{M/x\}$

# Rosser 1936, Plotkin 1975: Call-by-value

Value  $V, W ::= x \mid (V, W) \mid \text{inl } V \mid \text{inr } W \mid \lambda x. N$

$(\beta\&)$	$\text{fst } (V, W)$	$\longrightarrow_v$	$V$
$(\beta\&)$	$\text{snd } (V, W)$	$\longrightarrow_v$	$W$
$(\beta\vee)$	$\text{case inl } V \text{ of inl } x \Rightarrow P, \text{ inr } y \Rightarrow Q$	$\longrightarrow_v$	$P\{V/x\}$
$(\beta\vee)$	$\text{case inr } W \text{ of inl } x \Rightarrow P, \text{ inr } y \Rightarrow Q$	$\longrightarrow_v$	$Q\{W/y\}$
$(\beta\supset)$	$(\lambda x. N) V$	$\longrightarrow_v$	$N\{V/x\}$



## Part 6

# Sequent Calculus (Classical)

# Gentzen 1935: Sequent Calculus

$$\&-IS: \frac{\Gamma \rightarrow \Theta, \mathfrak{A} \quad \Gamma \rightarrow \Theta, \mathfrak{B}}{\Gamma \rightarrow \Theta, \mathfrak{A} \& \mathfrak{B}},$$

$$\&-IA: \frac{\mathfrak{A}, \Gamma \rightarrow \Theta}{\mathfrak{A} \& \mathfrak{B}, \Gamma \rightarrow \Theta} \quad \frac{\mathfrak{B}, \Gamma \rightarrow \Theta}{\mathfrak{A} \& \mathfrak{B}, \Gamma \rightarrow \Theta},$$

$$\vee-IA: \frac{\mathfrak{A}, \Gamma \rightarrow \Theta \quad \mathfrak{B}, \Gamma \rightarrow \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \rightarrow \Theta},$$

$$\vee-IS: \frac{\Gamma \rightarrow \Theta, \mathfrak{A}}{\Gamma \rightarrow \Theta, \mathfrak{A} \vee \mathfrak{B}} \quad \frac{\Gamma \rightarrow \Theta, \mathfrak{B}}{\Gamma \rightarrow \Theta, \mathfrak{A} \vee \mathfrak{B}},$$

$$\forall-IS: \frac{\Gamma \rightarrow \Theta, \mathfrak{A}}{\Gamma \rightarrow \Theta, \forall x \mathfrak{A}},$$

$$\exists-IA: \frac{\mathfrak{A}, \Gamma \rightarrow \Theta}{\exists x \mathfrak{A}, \Gamma \rightarrow \Theta}.$$

## Gentzen 1935: Logical rules

$$\frac{\Gamma \rightarrow \Theta, A \quad \Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \& B} \&R$$

$$\frac{A, \Gamma \rightarrow \Theta}{A \& B, \Gamma \rightarrow \Theta} \quad \frac{B, \Gamma \rightarrow \Theta}{A \& B, \Gamma \rightarrow \Theta} \&L$$

$$\frac{\Gamma \rightarrow \Theta, A}{\Gamma \rightarrow \Theta, A \vee B} \quad \frac{\Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \vee B} \vee R$$

$$\frac{A, \Gamma \rightarrow \Theta \quad B, \Gamma \rightarrow \Theta}{A \vee B, \Gamma \rightarrow \Theta} \vee L$$

$$\frac{A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, \neg A} \neg R \quad \frac{\Gamma \rightarrow \Theta, A}{\neg A, \Gamma \rightarrow \Theta} \neg L$$

## Gentzen 1935: Structural rules

$$\frac{}{A \rightarrow A} \text{Id}$$

$$\frac{\Gamma \rightarrow \Theta, A \quad A, \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda} \text{Cut}$$

## Gentzen 1935: Duality

$$(X)^\circ \equiv X$$

$$(A \& B)^\circ \equiv A^\circ \vee B^\circ$$

$$(A \vee B)^\circ \equiv A^\circ \& B^\circ$$

$$(\neg A)^\circ \equiv \neg A^\circ$$

$$(A_1, \dots, A_n)^\circ \equiv A_n^\circ, \dots, A_1^\circ$$

**Proposition 1** *A sequent is derivable if and only if its dual is derivable,*

$$\Gamma \rightarrow \Theta \quad \text{iff} \quad \Theta^\circ \rightarrow \Gamma^\circ.$$

# Gentzen 1935: Cut Elimination

$$\frac{\frac{\frac{\Gamma \rightarrow \Theta, A \quad \Gamma \rightarrow \Theta, B}{\Gamma \rightarrow \Theta, A \& B} \&R \quad \frac{A, \Gamma \rightarrow \Theta}{A \& B, \Gamma \rightarrow \Theta} \&L}{\Gamma \rightarrow \Theta} \text{Cut}}$$

$$\longrightarrow \frac{\Gamma \rightarrow \Theta, A \quad A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta} \text{Cut}$$

$$\frac{\frac{\frac{A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, \neg A} \neg R \quad \frac{\Gamma \rightarrow \Theta, A}{\neg A, \Gamma \rightarrow \Theta} \neg L}{\Gamma \rightarrow \Theta} \text{Cut}}$$

$$\longrightarrow \frac{\Gamma \rightarrow \Theta, A \quad A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta} \text{Cut}$$

## Part 7

# The dual calculus

# Intuitionistic natural deduction

Term  $x_1 : A_1, \dots, x_m : A_m \rightarrow M : A$

# Classical sequent calculus

Term  $x_1 : A_1, \dots, x_m : A_m \rightarrow \alpha_1 : B_1, \dots, \alpha_n : B_n \mid M : A$

Coterm  $K : A \mid x_1 : A_1, \dots, x_m : A_m \rightarrow \alpha_1 : B_1, \dots, \alpha_n : B_n$

Statement  $x_1 : A_1, \dots, x_m : A_m \mid S \vdash \alpha_1 : B_1, \dots, \alpha_n : B_n$



# Terms, Coterms, Statements

Term	$M, N$	$::=$	$x \mid \langle M, N \rangle \mid \langle M \rangle \text{inl} \mid \langle N \rangle \text{inr} \mid [K] \text{not} \mid (S).\alpha$
Coterm	$K, L$	$::=$	$\alpha \mid [K, L] \mid \text{fst}[K] \mid \text{snd}[L] \mid \text{not} \langle M \rangle \mid x.(S)$
Statement	$S, T$	$::=$	$M \bullet K$

Right sequent  $\Gamma \rightarrow \Theta \mid M : A$

Left sequent  $K : A \mid \Gamma \rightarrow \Theta$

Center sequent  $\Gamma \mid S \vdash \Theta$

# Logical rules

$$\frac{\Gamma \rightarrow \Theta \mid M : A \quad \Gamma \rightarrow \Theta \mid N : B}{\Gamma \rightarrow \Theta \mid \langle M, N \rangle : A \& B} \&R$$

$$\frac{K : A \mid \Gamma \rightarrow \Theta}{\text{fst}[K] : A \& B \mid \Gamma \rightarrow \Theta} \quad \frac{L : B \mid \Gamma \rightarrow \Theta}{\text{snd}[L] : A \& B \mid \Gamma \rightarrow \Theta} \&L$$

$$\frac{\Gamma \rightarrow \Theta \mid M : A}{\Gamma \rightarrow \Theta \mid \langle M \rangle \text{inl} : A \vee B} \quad \frac{\Gamma \rightarrow \Theta \mid N : B}{\Gamma \rightarrow \Theta \mid \langle N \rangle \text{inr} : A \vee B} \vee R$$

$$\frac{K : A \mid \Gamma \rightarrow \Theta \quad L : B \mid \Gamma \rightarrow \Theta}{[K, L] : A \vee B \mid \Gamma \rightarrow \Theta} \vee L$$

$$\frac{K : A \mid \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta \mid [K] \text{not} : \neg A} \neg R$$

$$\frac{\Gamma \rightarrow \Theta \mid M : A}{\text{not} \langle M \rangle : \neg A \mid \Gamma \rightarrow \Theta} \neg L$$

# Structural rules

$$\frac{}{x:A \rightarrow \mathbf{!}x:A} \text{IdR}$$

$$\frac{}{\alpha:A \mathbf{!} \rightarrow \alpha:A} \text{IdL}$$

$$\frac{\Gamma \mathbf{!} S \vdash \Theta, \alpha:A}{\Gamma \rightarrow \Theta \mathbf{!} (S).\alpha:A} \text{RI}$$

$$\frac{x:A, \Gamma \mathbf{!} S \vdash \Theta}{x.(S):A \mathbf{!} \Gamma \rightarrow \Theta} \text{LI}$$

$$\frac{\Gamma \rightarrow \Theta \mathbf{!} M:A \quad K:A \mathbf{!} \Delta \rightarrow \Lambda}{\Gamma, \Delta \mathbf{!} M \bullet K \vdash \Theta, \Lambda} \text{Cut}$$

# Duality

$$(X)^\circ \equiv X$$

$$(A \& B)^\circ \equiv A^\circ \vee B^\circ$$

$$(A \vee B)^\circ \equiv A^\circ \& B^\circ$$

$$(\neg A)^\circ \equiv \neg A^\circ$$

$$(x)^\circ \equiv x^\circ$$

$$(\alpha)^\circ \equiv \alpha^\circ$$

$$(\langle M, N \rangle)^\circ \equiv [M^\circ, N^\circ]$$

$$([K, L])^\circ \equiv \langle K^\circ, L^\circ \rangle$$

$$(\langle M \rangle \text{inl})^\circ \equiv \text{fst}[M^\circ]$$

$$(\text{fst}[K])^\circ \equiv \langle K^\circ \rangle \text{inl}$$

$$(\langle N \rangle \text{inr})^\circ \equiv \text{snd}[M^\circ]$$

$$(\text{snd}[L])^\circ \equiv \langle K^\circ \rangle \text{inr}$$

$$([K] \text{not})^\circ \equiv \text{not} \langle K^\circ \rangle$$

$$(\text{not} \langle M \rangle)^\circ \equiv [M^\circ] \text{not}$$

$$((S).\alpha)^\circ \equiv \alpha^\circ.(S^\circ)$$

$$(x.(S))^\circ \equiv (S^\circ).x^\circ$$

$$(M \bullet K)^\circ \equiv K^\circ \bullet M^\circ$$

# Duality

**Proposition 2** *A sequent is derivable if and only if its dual is derivable,*

$$\left. \begin{array}{l} \Gamma \rightarrow \Theta \mid M : A \\ K : A \mid \Gamma \rightarrow \Theta \\ \Gamma \mid S \vdash \Theta \end{array} \right\} \text{iff} \left\{ \begin{array}{l} M^\circ : A^\circ \mid \Theta^\circ \rightarrow \Gamma^\circ \\ \Theta^\circ \rightarrow \Gamma^\circ \mid K^\circ : A^\circ \\ \Theta^\circ \mid S^\circ \vdash \Gamma^\circ. \end{array} \right.$$

# Gentzen (1935): Cut Elimination

$$\frac{\frac{\Gamma \rightarrow \Theta \mid M : A \quad \Gamma \rightarrow \Theta \mid N : B}{\Gamma \rightarrow \Theta \mid \langle M, N \rangle : A \& B} \&R \quad \frac{K : A \mid \Gamma \rightarrow \Theta}{\text{fst}[K] : A \& B \mid \Gamma \rightarrow \Theta} \&L}{\Gamma \mid \langle M, N \rangle \bullet \text{fst}[K] \Vdash \Theta} \text{Cut}$$

$$\longrightarrow \frac{\Gamma \rightarrow \Theta \mid M : A \quad K : A \mid \Gamma \rightarrow \Theta}{\Gamma \mid M \bullet K \Vdash \Theta} \text{Cut}$$

$$\frac{\frac{K : A \mid \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta \mid [K]\text{not} : \neg A} \neg R \quad \frac{\Gamma \rightarrow \Theta \mid M : A}{\text{not}\langle M \rangle : \neg A \mid \Gamma \rightarrow \Theta} \neg L}{\Gamma \mid [K]\text{not} \bullet \text{not}\langle M \rangle \Vdash \Theta} \text{Cut}$$

$$\longrightarrow \frac{\Gamma \rightarrow \Theta \mid M : A \quad K : A \mid \Gamma \rightarrow \Theta}{\Gamma \mid M \bullet K \Vdash \Theta} \text{Cut}$$

Part 8

Excluded middle

# Excluded middle

$$\begin{array}{c}
 \frac{}{x : A \rightarrow \mathbf{I} x : A} \text{IdR} \\
 \frac{}{x : A \rightarrow \mathbf{I} \langle x \rangle \text{inl} : A \vee \neg A} \vee\text{R} \\
 \frac{}{x : A \mathbf{I} \langle x \rangle \text{inl} \bullet \gamma \vdash \gamma : A \vee \neg A} \text{RE} \\
 \frac{}{x.(\langle x \rangle \text{inl} \bullet \gamma) : A \mathbf{I} \rightarrow \gamma : A \vee \neg A} \text{LI} \\
 \frac{}{\rightarrow \gamma : A \vee \neg A \mathbf{I} [x.(\langle x \rangle \text{inl} \bullet \gamma)] \text{not} : \neg A} \neg\text{R} \\
 \frac{}{\rightarrow \gamma : A \vee \neg A \mathbf{I} \langle [x.(\langle x \rangle \text{inl} \bullet \gamma)] \text{not} \rangle \text{inr} : A \vee \neg A} \vee\text{R} \\
 \frac{}{\mathbf{I} \langle [x.(\langle x \rangle \text{inl} \bullet \gamma)] \text{not} \rangle \text{inr} \bullet \delta \vdash \gamma : A \vee \neg A, \delta : A \vee \neg A} \text{RE} \\
 \frac{}{\mathbf{I} \langle [x.(\langle x \rangle \text{inl} \bullet \gamma)] \text{not} \rangle \text{inr} \bullet \gamma \vdash \gamma : A \vee \neg A} \text{Cont} \\
 \frac{}{\rightarrow \mathbf{I} (\langle [x.(\langle x \rangle \text{inl} \bullet \gamma)] \text{not} \rangle \text{inr} \bullet \gamma). \gamma : A \vee \neg A} \text{RI}
 \end{array}$$



## Part 9

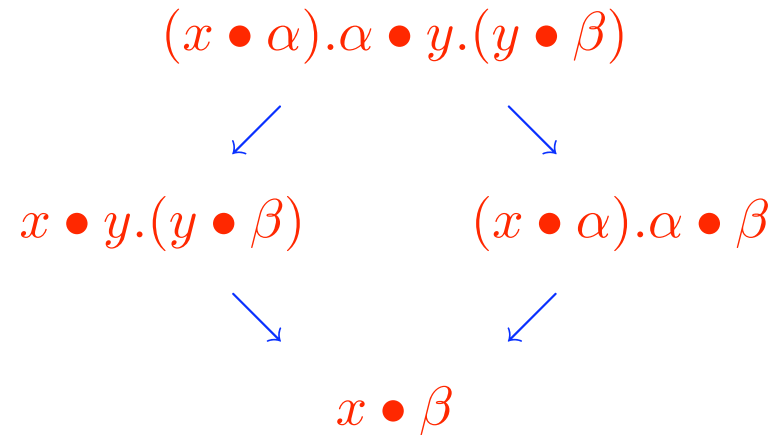
Call-by-value is Dual to Call-by-name

# Critical pair

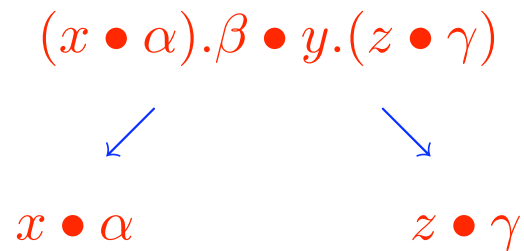
$$(\beta\text{L}) \quad M \bullet x.(S) \longrightarrow S\{M/x\}$$

$$(\beta\text{R}) \quad (S).\alpha \bullet K \longrightarrow S\{K/\alpha\}$$

Sometimes confluent.



Sometimes not.



# Call-by-value

$$(\beta\text{L}) \quad V \bullet x.(S) \longrightarrow S\{V/x\}$$

$$(\beta\text{R}) \quad (S).\alpha \bullet K \longrightarrow S\{K/\alpha\}$$

Sometimes confluent.

$$\begin{array}{c} (x \bullet \alpha).\alpha \bullet y.(y \bullet \beta) \\ \swarrow \\ x \bullet y.(y \bullet \beta) \\ \searrow \\ x \bullet \beta \end{array}$$

Sometimes not.

$$\begin{array}{c} (x \bullet \alpha).\beta \bullet y.(z \bullet \gamma) \\ \swarrow \\ x \bullet \alpha \end{array}$$

# Call-by-name

$$(\beta\text{L}) \quad M \bullet x.(S) \longrightarrow S\{M/x\}$$

$$(\beta\text{R}) \quad (S).\alpha \bullet P \longrightarrow S\{P/\alpha\}$$

Sometimes confluent.

$$\begin{array}{c} (x \bullet \alpha).\alpha \bullet y.(y \bullet \beta) \\ \searrow \\ (x \bullet \alpha).\alpha \bullet \beta \\ \swarrow \\ x \bullet \beta \end{array}$$

Sometimes not.

$$\begin{array}{c} (x \bullet \alpha).\beta \bullet y.(z \bullet \gamma) \\ \searrow \\ z \bullet \gamma \end{array}$$

# Call-by-value reductions

Value  $V, W ::= x \mid \langle V, W \rangle \mid \langle V \rangle \text{inl} \mid \langle W \rangle \text{inr} \mid [K] \text{not}$

$$(\beta\&) \quad \langle V, W \rangle \bullet \text{fst}[K] \longrightarrow_v V \bullet K$$

$$(\beta\&) \quad \langle V, W \rangle \bullet \text{snd}[L] \longrightarrow_v W \bullet L$$

$$(\beta\vee) \quad \langle V \rangle \text{inl} \bullet [K, L] \longrightarrow_v V \bullet K$$

$$(\beta\vee) \quad \langle W \rangle \text{inr} \bullet [K, L] \longrightarrow_v W \bullet L$$

$$(\beta\neg) \quad [K] \text{not} \bullet \text{not} \langle M \rangle \longrightarrow_v M \bullet K$$

$$(\beta\text{L}) \quad V \bullet x.(S) \longrightarrow_v S\{V/x\}$$

$$(\beta\text{R}) \quad (S).\alpha \bullet K \longrightarrow_v S\{K/\alpha\}$$

# Call-by-name reductions

Covalue  $P, Q ::= \alpha \mid [P, Q] \mid \text{fst}[P] \mid \text{snd}[Q] \mid \text{not}\langle M \rangle$

$$(\beta\vee) \quad \langle M \rangle \text{inl} \bullet [P, Q] \longrightarrow_n M \bullet P$$

$$(\beta\vee) \quad \langle N \rangle \text{inr} \bullet [P, Q] \longrightarrow_n N \bullet Q$$

$$(\beta\&) \quad \langle M, N \rangle \bullet \text{fst}[P] \longrightarrow_n M \bullet P$$

$$(\beta\&) \quad \langle M, N \rangle \bullet \text{snd}[Q] \longrightarrow_n N \bullet Q$$

$$(\beta\neg) \quad [K] \text{not} \bullet \text{not}\langle M \rangle \longrightarrow_n M \bullet K$$

$$(\beta R) \quad (S).\alpha \bullet P \longrightarrow_n S\{P/\alpha\}$$

$$(\beta L) \quad M \bullet x.(S) \longrightarrow_n S\{M/x\}$$

# Call-by-value, additional reductions

Term context  $E ::= \langle \{ \}, M \rangle \mid \langle V, \{ \} \rangle \mid \langle \{ \} \rangle_{\text{inl}} \mid \langle \{ \} \rangle_{\text{inr}}$

$$(\eta\text{L}) \quad K \quad \longrightarrow_v \quad x.(x \bullet K)$$

$$(\eta\text{R}) \quad M \quad \longrightarrow_v \quad (M \bullet \alpha).\alpha$$

$$(\zeta) \quad E\{M\} \quad \longrightarrow_v \quad (M \bullet x.(E\{x\} \bullet \beta)).\beta$$

# Call-by-name, additional reductions

Coterm context  $F ::= [\{\}, K] \mid [P, \{\}] \mid \text{fst}[\{\}] \mid \text{snd}[\{\}]$

$$(\eta\text{R}) \quad M \longrightarrow_n (M \bullet \alpha).\alpha$$

$$(\eta\text{L}) \quad K \longrightarrow_n x.(x \bullet K)$$

$$(\varsigma) \quad F\{K\} \longrightarrow_n y.((y \bullet F\{\alpha\}).\alpha \bullet K)$$



# Call-by-value is dual to call-by-name

**Proposition 3** *Call-by-value is dual to call-by-name,*

$$\left. \begin{array}{l} M \longrightarrow_v N \\ K \longrightarrow_v L \\ S \longrightarrow_v T \end{array} \right\} \text{iff} \left\{ \begin{array}{l} M^\circ \longrightarrow_n N^\circ \\ K^\circ \longrightarrow_n L^\circ \\ S^\circ \longrightarrow_n T^\circ. \end{array} \right.$$

Part 10

Functions

# Encoding functions in call-by-value

$$\frac{x : A, \Gamma \rightarrow \Theta \mid N : B}{\Gamma \rightarrow \Theta \mid \lambda x. N : A \supset B} \supset R \qquad \frac{\Gamma \rightarrow \Theta \mid M : A \quad L : B \mid \Gamma \rightarrow \Theta}{M @ L : A \supset B \mid \Gamma \rightarrow \Theta} \supset L$$

$$(\beta \supset) \quad \lambda x. N \bullet V @ L \longrightarrow_v V \bullet x.(N \bullet L)$$

**Proposition 4** *Under call-by-value, implication can be defined by*

$$A \supset B \equiv \neg(A \& \neg B)$$

$$\lambda x. N \equiv [z.(z \bullet \text{fst}[x.(z \bullet \text{snd}[\text{not}\langle N \rangle])])]\text{not}$$

$$M @ L \equiv \text{not}\langle\langle M, [L]\text{not}\rangle\rangle.$$

*Note translation of a function abstraction is a value.*

# Encoding functions in call-by-name

$$\frac{x : A, \Gamma \rightarrow \Theta \mid N : B}{\Gamma \rightarrow \Theta \mid \lambda x. N : A \supset B} \supset R \qquad \frac{\Gamma \rightarrow \Theta \mid M : A \quad L : B \mid \Gamma \rightarrow \Theta}{M @ L : A \supset B \mid \Gamma \rightarrow \Theta} \supset L$$

$$(\beta \supset) \quad \lambda x. N \bullet M @ Q \longrightarrow_v M \bullet x.(N \bullet Q)$$

**Proposition 5** *Under call-by-name, implication can be defined by*

$$\begin{aligned} A \supset B &\equiv \neg A \vee B \\ \lambda x. N &\equiv (\langle [x. (\langle N \rangle \text{inr} \bullet \gamma)] \text{not} \rangle \text{inl} \bullet \gamma). \gamma \\ M @ L &\equiv [\text{not} \langle M \rangle, L]. \end{aligned}$$

*The translation of a function application is a covalue.*

## Part 11

# Continuation-passing style

# Call-by-value CPS

$$(X)^V \equiv X$$

$$(A \& B)^V \equiv (A)^V \times (B)^V$$

$$(A \vee B)^V \equiv (A)^V + (B)^V$$

$$(\neg A)^V \equiv (A)^V \rightarrow R$$

# Call-by-value CPS

$$\begin{aligned}(x)^v &\equiv \lambda\gamma. \gamma x \\ (\langle M, N \rangle)^v &\equiv \lambda\gamma. (M)^v (\lambda x. (N)^v (\lambda y. \gamma \langle x, y \rangle)) \\ (\langle M \rangle \text{inl})^v &\equiv \lambda\gamma. (M)^v (\lambda x. \gamma (\text{inl } x)) \\ (\langle N \rangle \text{inr})^v &\equiv \lambda\gamma. (N)^v (\lambda y. \gamma (\text{inr } y)) \\ ([K] \text{not})^v &\equiv \lambda\gamma. \gamma (\lambda z. (K)^v z) \\ ((S).\alpha)^v &\equiv \lambda\alpha. (S)^v \\ (\alpha)^v &\equiv \lambda z. \alpha z \\ ([K, L])^v &\equiv \lambda z. \text{case } z \text{ of inl } x \Rightarrow (K)^v x, \text{ inr } y \Rightarrow (L)^v y \\ (\text{fst}[K])^v &\equiv \lambda z. \text{case } z \text{ of } \langle x, - \rangle \Rightarrow (K)^v x \\ (\text{snd}[L])^v &\equiv \lambda z. \text{case } z \text{ of } \langle -, y \rangle \Rightarrow (L)^v y \\ (\text{not}\langle M \rangle)^v &\equiv \lambda z. (\lambda\gamma. (M)^v \gamma) z \\ (x.(S))^v &\equiv \lambda x. (S)^v \\ (M \bullet K)^v &\equiv (M)^v (K)^v\end{aligned}$$

# CPS preserves types

**Proposition 6** *The call-by-value CPS translation preserves types.*

$$\left. \begin{array}{l} \Gamma \rightarrow \Theta \mid V : A \\ \Gamma \rightarrow \Theta \mid M : A \\ K : A \mid \Gamma \rightarrow \Theta \\ \Gamma \mid S \mapsto \Theta \end{array} \right\} \text{iff} \left\{ \begin{array}{l} (\Gamma)^V, (\neg\Theta)^V \rightarrow (V)^V : (A)^V \\ (\Gamma)^V, (\neg\Theta)^V \rightarrow (M)^v : (\neg\neg A)^V \\ (\Gamma)^V, (\neg\Theta)^V \rightarrow (K)^v : (\neg A)^V \\ (\Gamma)^V, (\neg\Theta)^V \rightarrow (S)^v : R \end{array} \right.$$



## CPS preserves and reflect reductions

**Proposition 7** *Let  $M, K, S$  be in the dual calculus, and  $N, L, T$  be in the CPS target calculus. Then*

$$\left. \begin{array}{l} M \longrightarrow_v (N)_v \\ K \longrightarrow_v (L)_v \\ S \longrightarrow_v (T)_v \end{array} \right\} \text{iff} \left\{ \begin{array}{ll} (M)^v \longrightarrow N & ((N)_v)^v \equiv N \\ (K)^v \longrightarrow L & \text{and } ((L)_v)^v \equiv L \\ (S)^v \longrightarrow T, & ((T)_v)^v \equiv T. \end{array} \right.$$

# Call-by-name CPS

$$(X)^N \equiv X$$

$$(A \& B)^N \equiv (A)^N + (B)^N$$

$$(A \vee B)^N \equiv (A)^N \times (B)^N$$

$$(\neg A)^N \equiv (A)^N \rightarrow R$$

# Call-by-name CPS

$$\begin{aligned}(\alpha)^n &\equiv \lambda z. z \alpha \\([K, L])^n &\equiv \lambda z. (K)^n (\lambda \alpha. (L)^n (\lambda \beta. z \langle \alpha, \beta \rangle)) \\(\text{fst}[K])^n &\equiv \lambda z. (K)^n (\lambda \alpha. z (\text{inl } \alpha)) \\(\text{snd}[L])^n &\equiv \lambda z. (L)^n (\lambda \beta. z (\text{inr } \beta)) \\(\text{not}\langle M \rangle)^n &\equiv \lambda z. z (\lambda \gamma. (M)^n \gamma) \\(x.(S))^n &\equiv \lambda x. (S)^n \\(x)^n &\equiv \lambda \gamma. x \gamma \\(\langle M, N \rangle)^n &\equiv \lambda \gamma. \text{case } \gamma \text{ of inl } \alpha \Rightarrow (M)^n \alpha, \text{ inr } \beta \Rightarrow (N)^n \beta \\(\langle M \rangle \text{inl})^n &\equiv \lambda \gamma. \text{case } \gamma \text{ of } \langle \alpha, - \rangle \Rightarrow (M)^n \alpha \\(\langle N \rangle \text{inr})^n &\equiv \lambda \gamma. \text{case } \gamma \text{ of } \langle -, \beta \rangle \Rightarrow (N)^n \beta \\([K] \text{not})^n &\equiv \lambda \gamma. (\lambda z. (K)^n z) \gamma \\((S).\alpha)^n &\equiv \lambda \alpha. (S)^n \\(M \bullet K)^n &\equiv (K)^n (M)^n\end{aligned}$$

# CPS preserves types

**Proposition 8** *The call-by-name CPS translation preserves types.*

$$\left. \begin{array}{l} P : A \mid \Gamma \rightarrow \Theta \\ \Gamma \rightarrow \Theta \mid M : A \\ K : A \mid \Gamma \rightarrow \Theta \\ \Gamma \mid S \mapsto \Theta \end{array} \right\} \text{iff} \left\{ \begin{array}{l} (\neg\Gamma)^N, (\Theta)^N \rightarrow (P)^N : (A)^N \\ (\neg\Gamma)^N, (\Theta)^N \rightarrow (M)^n : (\neg A)^N \\ (\neg\Gamma)^N, (\Theta)^N \rightarrow (K)^n : (\neg\neg A)^N \\ (\neg\Gamma)^N, (\Theta)^N \rightarrow (S)^n : R \end{array} \right.$$

## CPS preserves and reflect reductions

**Proposition 9** *Let  $K, M, S$  be in the dual calculus, and  $L, N, T$  be in the CPS target calculus. Then*

$$\left. \begin{array}{l} M \longrightarrow_n (N)_n \\ K \longrightarrow_n (L)_n \\ S \longrightarrow_n (T)_n \end{array} \right\} \text{iff} \left\{ \begin{array}{ll} (M)^n \longrightarrow N & ((N)_n)^n \equiv N \\ (K)^n \longrightarrow L & \text{and } ((L)_n)^n \equiv L \\ (S)^n \longrightarrow T, & ((T)_n)^n \equiv T. \end{array} \right.$$

Part 12

Conclusions

# Contributions of this work

- Replace one connective by three:

$$A \supset B \quad \Longrightarrow \quad \begin{array}{l} A \& B \\ A \vee B \\ \neg A. \end{array}$$

- Encodings:

$$\begin{array}{ll} A \supset B \equiv (\neg A) \vee B, & \text{call-by-name,} \\ A \supset B \equiv \neg(A \& (\neg B)), & \text{call-by-value.} \end{array}$$

- Role of values and covalues clarified.
- Improved CPS results, following Sabry and Wadler (1997).

# Implementation

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## Another question

What more will we discover by the centenary of the birth of  $\lambda$ -calculus, natural deduction, and sequent calculus?